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FAST TRACK COMMUNICATION

Kerr–NUT–de Sitter curvature in all dimensions

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Abstract
We explicitly calculate the Riemannian curvature of D-dimensional metrics recently discussed by Chen, Lü and Pope. We find that it can be concisely written by using a single function. The Einstein condition which corresponds to the Kerr–NUT–de Sitter metric is clarified for all dimensions. It is shown that the metrics are of type D.

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1. Introduction

In recent years, studies of exact solutions to the higher dimensional Einstein equations have attracted much attention in the context of supergravity and superstring theories [1–6]. Here, we revisit a class of D-dimensional metrics discussed by Chen, Lü and Pope [6]:

(a) $D = 2n$

$$g = \sum_{\mu=1}^{n} \frac{\mathrm{d}x_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)}\, \mathrm{d}\psi_k \right)^2, \tag{1.1}$$

(b) $D = 2n + 1$

$$g = \sum_{\mu=1}^{n} \frac{\mathrm{d}x_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)}\, \mathrm{d}\psi_k \right)^2 + S \left(\sum_{k=0}^{n} A^{(k)}\, \mathrm{d}\psi_k \right)^2. \tag{1.2}$$

The functions $Q_\mu (\mu = 1, 2, \ldots, n)$ are given by

$$Q_\mu = \frac{X_\mu}{U_\mu}, \qquad U_\mu = \prod_{\substack{\nu=1 \\ (\nu\neq\mu)}}^{n} \left(x_\mu^2 - x_\nu^2\right), \tag{1.3}$$

where X_μ is an arbitrary function depending only on x_μ. The remaining functions are

$$A_\mu^{(k)} = \sum_{\substack{1\leqslant\nu_1<\nu_2<\cdots<\nu_k\leqslant n\\(\nu_i\neq\mu)}} x_{\nu_1}^2 x_{\nu_2}^2\cdots x_{\nu_k}^2, \qquad A^{(k)} = \sum_{1\leqslant\nu_1<\nu_2<\cdots<\nu_k\leqslant n} x_{\nu_1}^2 x_{\nu_2}^2\cdots x_{\nu_k}^2, \tag{1.4}$$

$\left(A_\mu^{(0)} = A^{(0)} = 1\right)$ and $S = c/A^{(n)}$ with a constant c. It has been shown by means of computer calculation that the metrics satisfy the Einstein equations $Ric(g) = \Lambda g$ for dimensions $D \leqslant 15$ if X_μ takes the form

(a) $D = 2n$

$$X_\mu = \sum_{k=0}^{n} c_{2k} x_\mu^{2k} + b_\mu x_\mu, \tag{1.5}$$

(b) $D = 2n + 1$

$$X_\mu = \sum_{k=1}^{n} c_{2k} x_\mu^{2k} + b_\mu + \frac{(-1)^n c}{x_\mu^2}, \tag{1.6}$$

where c, c_{2k} and b_μ are free parameters. This class of metrics gives the Kerr–NUT–de Sitter metric [6], and the solutions in [1–5] are recovered by choosing special parameters. However, the explicit Riemannian curvature was not given in their analysis. So it is obscure how the metrics become solutions to the Einstein equations. In this paper we give a systematic investigation of the Riemannian curvature. We show that it can be concisely written by using a single function. We also prove that metrics (1.1) and (1.2) are of type D and under conditions (1.5) and (1.6) they become Einstein metrics for all dimensions. This family of metrics is also interesting from the point of view of AdS/CFT correspondence. Indeed, odd-dimensional Einstein metrics lead to Sasaki–Einstein metrics by taking the BPS limit [6–9] and even-dimensional Einstein metrics lead to Calabi–Yau metrics in the limit [6, 10, 11]. Especially, the five-dimensional Sasaki–Einstein metrics have emerged quite naturally in the AdS/CFT correspondence.

2. $D = 2n$

For metric (1.1), we introduce the following orthonormal frame $\{e^a\} = \{e^\mu, e^{n+\mu}\}(\mu = 1, 2, \ldots, n)$:

$$e^\mu = \frac{\mathrm{d}x_\mu}{\sqrt{Q_\mu}}, \qquad e^{n+\mu} = \sqrt{Q_\mu}\left(\sum_{k=0}^{n-1} A_\mu^{(k)}\,\mathrm{d}\psi_k\right). \tag{2.1}$$

Using the first structure equation

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0 \tag{2.2}$$

and $\omega_{ab} = -\omega_{ba}$, we obtain connection 1-forms ω_{ab}. A straightforward calculation gives

$$\omega_{\mu\nu} = (1-\delta_{\mu\nu})\left[-\frac{x_\nu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2}e^\mu - \frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2}e^\nu\right],$$

$$\omega_{\mu,n+\nu} = \delta_{\mu\nu}\left[-\frac{\partial(\sqrt{Q_\mu})}{\partial x_\mu}e^{n+\mu} + \sum_{\substack{\rho=1\\(\rho\neq\mu)}}^{n} \frac{x_\mu\sqrt{Q_\rho}}{x_\mu^2 - x_\rho^2}e^{n+\rho}\right]$$

$$
+(1-\delta_{\mu\nu})\left[-\frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2-x_\nu^2}e^{n+\nu}+\frac{x_\mu\sqrt{Q_\nu}}{x_\mu^2-x_\nu^2}e^{n+\mu}\right],
$$

$$
\omega_{n+\mu,n+\nu}=(1-\delta_{\mu\nu})\left[-\frac{x_\mu\sqrt{Q_\nu}}{x_\mu^2-x_\nu^2}e^{\mu}-\frac{x_\nu\sqrt{Q_\mu}}{x_\mu^2-x_\nu^2}e^{\nu}\right]. \tag{2.3}
$$

From the second structure equation

$$
R_{ab}=\mathrm{d}\omega_{ab}+\omega_{ac}\wedge\omega^c{}_b, \tag{2.4}
$$

we can calculate the curvature 2-forms R_{ab}. It is convenient to introduce a quantity

$$
Q_T=\sum_{\mu=1}^{n}Q_\mu. \tag{2.5}
$$

We find $(\mu\neq\nu)$

$$
\begin{aligned}
R_{\mu\nu}&=-\frac{1}{2\left(x_\mu^2-x_\nu^2\right)}\left(x_\mu\frac{\partial Q_T}{\partial x_\mu}-x_\nu\frac{\partial Q_T}{\partial x_\nu}\right)e^\mu\wedge e^\nu\\
&\quad-\frac{1}{2\left(x_\mu^2-x_\nu^2\right)}\left(x_\nu\frac{\partial Q_T}{\partial x_\mu}-x_\mu\frac{\partial Q_T}{\partial x_\nu}\right)e^{n+\mu}\wedge e^{n+\nu},\\
R_{\mu,n+\mu}&=-\frac{1}{2}\frac{\partial^2 Q_T}{\partial x_\mu^2}e^\mu\wedge e^{n+\mu}\\
&\quad+\sum_{\rho\neq\mu}\frac{1}{x_\mu^2-x_\rho^2}\left(x_\mu\frac{\partial Q_T}{\partial x_\rho}-x_\rho\frac{\partial Q_T}{\partial x_\mu}\right)e^\rho\wedge e^{n+\rho},\qquad\text{(no sum)}\\
R_{\mu,n+\nu}&=-\frac{1}{2\left(x_\mu^2-x_\nu^2\right)}\left(x_\mu\frac{\partial Q_T}{\partial x_\mu}-x_\nu\frac{\partial Q_T}{\partial x_\nu}\right)e^\mu\wedge e^{n+\nu}\\
&\quad+\frac{1}{2\left(x_\mu^2-x_\nu^2\right)}\left(x_\mu\frac{\partial Q_T}{\partial x_\nu}-x_\nu\frac{\partial Q_T}{\partial x_\mu}\right)e^\nu\wedge e^{n+\mu},\\
R_{n+\mu,n+\nu}&=-\frac{1}{2\left(x_\mu^2-x_\nu^2\right)}\left(x_\nu\frac{\partial Q_T}{\partial x_\mu}-x_\mu\frac{\partial Q_T}{\partial x_\nu}\right)e^\mu\wedge e^\nu\\
&\quad-\frac{1}{2\left(x_\mu^2-x_\nu^2\right)}\left(x_\mu\frac{\partial Q_T}{\partial x_\mu}-x_\nu\frac{\partial Q_T}{\partial x_\nu}\right)e^{n+\mu}\wedge e^{n+\nu}.
\end{aligned} \tag{2.6}
$$

Let I_μ be the differential operator

$$
I_\mu=\frac{1}{2}\frac{\partial^2}{\partial x_\mu^2}+\sum_{\rho\neq\mu}\frac{1}{x_\rho^2-x_\mu^2}\left(x_\rho\frac{\partial}{\partial x_\rho}-x_\mu\frac{\partial}{\partial x_\mu}\right). \tag{2.7}
$$

The components $\mathcal{R}_{ab}=\sum_{c=1}^{2n}R^c{}_{acb}$ of the Ricci curvature are calculated as

$$
\mathcal{R}_{\mu\nu}=\mathcal{R}_{n+\mu,n+\nu}=-\delta_{\mu\nu}I_\mu(Q_T). \tag{2.8}
$$

Using the expression $Q_T=\sum_{\mu=1}^{n}(X_\mu/U_\mu)$, we have

$$
I_\mu(Q_T)=\frac{1}{2}\frac{X_\mu''}{U_\mu}+\sum_{\rho\neq\mu}\frac{1}{x_\rho^2-x_\mu^2}\left(x_\rho\frac{X_\rho'}{U_\rho}+x_\mu\frac{X_\mu'}{U_\mu}\right)-\sum_{\rho\neq\mu}\frac{1}{x_\rho^2-x_\mu^2}\left(\frac{X_\rho}{U_\rho}+\frac{X_\mu}{U_\mu}\right), \tag{2.9}
$$

where $X_\mu'=\mathrm{d}X_\mu/\mathrm{d}x_\mu$ and $X_\mu''=\mathrm{d}^2X_\mu\big/\mathrm{d}x_\mu^2$. Thus, the scalar curvature $\mathcal{R}=\sum_{a=1}^{2n}\mathcal{R}_{aa}$ takes the form

$$
\mathcal{R}=-\sum_{\mu=1}^{n}\frac{X_\mu''}{U_\mu}. \tag{2.10}
$$

2.1. Einstein condition

We first study a condition $\mathcal{R} = \text{const}$. By (2.10) and the identities

$$\sum_{\mu=1}^{n} \frac{x_\mu^{2(n-1)}}{U_\mu} = 1, \qquad \sum_{\mu=1}^{n} \frac{x_\mu^{2k}}{U_\mu} = 0 \tag{2.11}$$

($k = 0, 1, \ldots, n-2$), it is easy to see that the function

$$X_\mu = \sum_{k=1}^{n} c_{2k} x_\mu^{2k} + b_\mu x_\mu + d_\mu \tag{2.12}$$

gives a constant scalar curvature $\mathcal{R} = -2n(2n-1)c_{2n}$ for arbitrary constants c_{2k}, b_μ and d_μ. Conversely, we can show that the condition $\mathcal{R} = \text{const}$ implies (2.12). In fact, from (2.10) one has

$$\mathcal{R} U_\mu = -X_\mu'' - \sum_{\nu\neq\mu} P_\mu^\nu X_\nu'', \tag{2.13}$$

where

$$P_\mu^\nu = \frac{U_\mu}{U_\nu} = -\frac{\prod_{\sigma\neq\mu,\nu}\left(x_\mu^2 - x_\sigma^2\right)}{\prod_{\lambda\neq\mu,\nu}\left(x_\nu^2 - x_\lambda^2\right)}. \tag{2.14}$$

Applying the differential operator $(\partial/\partial x_\mu)^{2n-1}$ to this relation we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}x_\mu}\right)^{2n+1} X_\mu = 0, \tag{2.15}$$

which means that X_μ must be polynomials of order $2n$. Taking X_μ in the general polynomials of order $2n$ we infer from (2.13) that they have the form (2.12). Thus, we have shown that the scalar curvature is a constant if and only if X_μ takes the form (2.12).

Now, we can examine the Einstein condition, i.e. $\mathcal{R}_{ab} = \Lambda\delta_{ab}$, where Λ represents a cosmological constant. Substituting (2.12) into (2.9) we obtain

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{n+\mu,n+\nu} = \delta_{\mu\nu}(-(2n-1)c_{2n} + K_\mu), \tag{2.16}$$

where

$$K_\mu = \sum_{\rho\neq\mu} \frac{1}{x_\rho^2 - x_\mu^2}\left(\frac{d_\rho}{U_\rho} + \frac{d_\mu}{U_\mu}\right). \tag{2.17}$$

The Einstein condition requires $K_1 = K_2 = \cdots = K_n = \text{const}$. This implies $d_1 = d_2 = \cdots = d_n$, and then $K_\mu = 0$. Therefore, denoting the common value of d_μ by c_0 we reproduce the function X_μ given in (1.5). It should be noted that

$$Q_T = \sum_{\mu=1}^{n} \frac{X_\mu}{U_\mu} = c_{2n} \sum_{\mu=1}^{n} x_\mu^2 + c_{2n-2} + V, \tag{2.18}$$

where

$$V = \sum_{\mu=1}^{n} \frac{b_\mu x_\mu}{U_\mu}. \tag{2.19}$$

The Ricci curvature is given by

$$\mathcal{R}_{ab} = (2n-1)\lambda\delta_{ab}, \tag{2.20}$$

with $\lambda = -c_{2n}$. From (2.6) the corresponding curvature 2-forms are written as ($\mu \neq \nu$)

$$
\begin{aligned}
R_{\mu\nu} &= \left[\lambda - \frac{1}{2(x_\mu^2 - x_\nu^2)}\left(x_\mu \frac{\partial V}{\partial x_\mu} - x_\nu \frac{\partial V}{\partial x_\nu}\right)\right] e^\mu \wedge e^\nu \\
&\quad - \frac{1}{2(x_\mu^2 - x_\nu^2)}\left(x_\nu \frac{\partial V}{\partial x_\mu} - x_\mu \frac{\partial V}{\partial x_\nu}\right) e^{n+\mu} \wedge e^{n+\nu}, \\
R_{\mu,n+\mu} &= \left(\lambda - \frac{1}{2}\frac{\partial^2 V}{\partial x_\mu^2}\right) e^\mu \wedge e^{n+\mu} \\
&\quad + \sum_{\rho\neq\mu} \frac{1}{x_\mu^2 - x_\rho^2}\left(x_\mu \frac{\partial V}{\partial x_\rho} - x_\rho \frac{\partial V}{\partial x_\mu}\right) e^\rho \wedge e^{n+\rho}, \qquad \text{(no sum)} \\
R_{\mu,n+\nu} &= \left[\lambda - \frac{1}{2(x_\mu^2 - x_\nu^2)}\left(x_\mu \frac{\partial V}{\partial x_\mu} - x_\nu \frac{\partial V}{\partial x_\nu}\right)\right] e^\mu \wedge e^{n+\nu} \\
&\quad + \frac{1}{2(x_\mu^2 - x_\nu^2)}\left(x_\mu \frac{\partial V}{\partial x_\nu} - x_\nu \frac{\partial V}{\partial x_\mu}\right) e^\nu \wedge e^{n+\mu}, \\
R_{n+\mu,n+\nu} &= -\frac{1}{2(x_\mu^2 - x_\nu^2)}\left(x_\nu \frac{\partial V}{\partial x_\mu} - x_\mu \frac{\partial V}{\partial x_\nu}\right) e^\mu \wedge e^\nu \\
&\quad + \left[\lambda - \frac{1}{2(x_\mu^2 - x_\nu^2)}\left(x_\mu \frac{\partial V}{\partial x_\mu} - x_\nu \frac{\partial V}{\partial x_\nu}\right)\right] e^{n+\mu} \wedge e^{n+\nu}.
\end{aligned}
\tag{2.21}
$$

If we put $b_\mu = 0$ for all μ, then equations represent the constant curvature space, $R_{ab} = \lambda e^a \wedge e^b$.

2.2. *Kähler condition*

The natural Kähler form associated with metric (1.1) is

$$
\begin{aligned}
\omega &= \sum_{\mu=1}^{n} e^\mu \wedge e^{n+\mu} \\
&= \sum_{\mu=1}^{n} \mathrm{d}x_\mu \wedge \left(\sum_{k=0}^{n-1} A_\mu^{(k)}\, \mathrm{d}\psi_k\right).
\end{aligned}
\tag{2.22}
$$

This 2-form is not closed, but there exists a scaling limit in which it becomes closed. Indeed we can take a limit $x_\mu = 1 + \epsilon\xi_\mu (\epsilon \to 0)$ together with a suitable transformation of the coordinates ψ_k.[3] Then, we have a closed 2-form in the form $\omega = \sum_i^n \mathrm{d}\sigma_i \wedge \mathrm{d}t^i$, where σ_i are the elementary symmetric polynomials of ξ_μ s. Now metric (1.1) reduces to the Kähler metric presented in [13] (see proposition 11).

3. $D = 2n + 1$

For metric (1.2), we introduce the following orthonormal frame $\{\hat{e}^a\} = \{\hat{e}^\mu, \hat{e}^{n+\mu}, \hat{e}^{2n+1}\}$ ($\mu = 1, 2, \ldots, n$):

$$
\hat{e}^\mu = e^\mu, \qquad \hat{e}^{n+\mu} = e^{n+\mu}, \qquad \hat{e}^{2n+1} = \sqrt{S}\left(\sum_{k=0}^{n} A^{(k)}\, \mathrm{d}\psi_k\right), \tag{3.1}
$$

[3] The scaling limit in four dimension was explicitly given in [12].

where e^{μ} and $e^{n+\mu}$ are defined by (2.1). The connection 1-forms $\hat{\omega}_{ab}$ are given by

$$\begin{aligned}
\hat{\omega}_{\mu\nu} &= \omega_{\mu\nu}, \\
\hat{\omega}_{\mu,n+\nu} &= \omega_{\mu,n+\nu} + \delta_{\mu\nu}\frac{\sqrt{S}}{x_\mu}\hat{e}^{2n+1}, \\
\hat{\omega}_{n+\mu,n+\nu} &= \omega_{n+\mu,n+\nu}, \\
\hat{\omega}_{\mu,2n+1} &= \frac{\sqrt{S}}{x_\mu}\hat{e}^{n+\mu} - \frac{\sqrt{Q_\mu}}{x_\mu}\hat{e}^{2n+1}, \\
\hat{\omega}_{n+\mu,2n+1} &= -\frac{\sqrt{S}}{x_\mu}\hat{e}^{\mu},
\end{aligned} \tag{3.2}$$

with ω_{ab} defined by (2.3). Shifting the arbitrary function X_μ by

$$X_\mu = \hat{X}_\mu + \frac{(-1)^n c}{x_\mu^2}, \tag{3.3}$$

we have

$$Q_T = \hat{Q}_T - S, \qquad \hat{Q}_T = \sum_{\mu=1}^{n}\frac{\hat{X}_\mu}{U_\mu}. \tag{3.4}$$

Then, the curvature 2-forms $\hat{R}_{\mu\nu}$, $\hat{R}_{\mu,n+\nu}$ and $\hat{R}_{n+\mu,n+\nu}$ are obtained by the replacement $Q_T \to \hat{Q}_T$ in (2.6), and the remaining ones are calculated as

$$\hat{R}_{\mu,2n+1} = -\frac{1}{2x_\mu}\frac{\partial \hat{Q}_T}{\partial x_\mu}\hat{e}^{\mu}\wedge\hat{e}^{2n+1}, \qquad \hat{R}_{n+\mu,2n+1} = -\frac{1}{2x_\mu}\frac{\partial \hat{Q}_T}{\partial x_\mu}\hat{e}^{n+\mu}\wedge\hat{e}^{2n+1}. \tag{3.5}$$

The Ricci curvature $\hat{\mathcal{R}}_{ab}$ and the scalar curvature $\hat{\mathcal{R}}$ are given by

$$\hat{\mathcal{R}}_{\mu\nu} = \hat{\mathcal{R}}_{n+\mu,n+\nu} = -\delta_{\mu\nu}\left(I_\mu(\hat{Q}_T) + \frac{1}{2x_\mu}\frac{\partial \hat{Q}_T}{\partial x_\mu}\right), \qquad \hat{\mathcal{R}}_{2n+1,2n+1} = -\sum_{\rho}\frac{1}{x_\rho}\frac{\partial \hat{Q}_T}{\partial x_\rho} \tag{3.6}$$

and

$$\hat{\mathcal{R}} = -\sum_{\mu=1}^{n}\frac{\hat{X}''_\mu}{U_\mu} - 2\sum_{\mu=1}^{n}\frac{1}{x_\mu}\frac{\hat{X}'_\mu}{U_\mu}. \tag{3.7}$$

3.1. Einstein condition

Using similar arguments to the case of $D = 2n$, we can show that the scalar curvature is a constant if and only if X_μ takes of the form

$$X_\mu = \sum_{k=1}^{n} c_{2k}x_\mu^{2k} + b_\mu + \frac{d_\mu}{x_\mu} + \frac{(-1)^n c}{x_\mu^2}. \tag{3.8}$$

Then the components of the Ricci curvature are

$$\begin{aligned}
\hat{\mathcal{R}}_{\mu\nu} = \hat{\mathcal{R}}_{n+\mu,n+\nu} &= -\delta_{\mu\nu}\left(2nc_{2n} + \frac{1}{2}\frac{d_\mu}{x_\mu^3 U_\mu} - \sum_{\rho\neq\mu}\left(\frac{d_\rho}{x_\rho U_\rho} + \frac{d_\mu}{x_\mu U_\mu}\right)\right), \\
\hat{\mathcal{R}}_{2n+1,2n+1} &= -2nc_{2n} + \sum_{\mu=1}^{n}\frac{d_\mu}{x_\mu^3 U_\mu},
\end{aligned} \tag{3.9}$$

which satisfy the Einstein condition if and only if d_μ vanishes for all μ. Thus we reproduce the function X_μ given in (1.6). Now $\hat{\mathcal{R}}_{ab} = 2n\lambda\delta_{ab}$ with $\lambda = -c_{2n}$ and Q_T is given by

$$Q_T = c_{2n}\left(\sum_{\mu=1}^{n} x_\mu^2\right) + c_{2n-2} + \hat{V} - S, \tag{3.10}$$

where

$$\hat{V} = \sum_{\mu=1}^{n} \frac{b_\mu}{U_\mu}. \tag{3.11}$$

By substituting these expressions into (2.6) and (3.5), we obtain (2.21) replaced V with $\hat{V}$, and

$$\hat{R}_{\mu,2n+1} = \left(\lambda - \frac{1}{2x_\mu}\frac{\partial \hat{V}}{\partial x_\mu}\right) e^\mu \wedge \hat{e}^{2n+1}, \qquad \hat{R}_{n+\mu,2n+1} = \left(\lambda - \frac{1}{2x_\mu}\frac{\partial \hat{V}}{\partial x_\mu}\right) e^{n+\mu} \wedge \hat{e}^{2n+1}. \tag{3.12}$$

If we choose all of b_μ as an equal value, then $\hat{V} = 0$ and hence the equations represent the constant curvature space.

4. Concluding remarks

We have explicitly calculated the Riemannian curvature corresponding to metrics (1.1) and (1.2). The components have a compact expression by introducing the single function Q_T. We have also proved that conditions (1.5) and (1.6) lead to the Einstein metrics for all dimensions.

Finally, we comment on type D condition [14]. For fixed μ, let us define the complex vector fields[4]

$$k = Q_\mu^{-1/2}(e_\mu + \mathrm{i}e_{n+\mu})/\sqrt{2}, \qquad \ell = Q_\mu^{1/2}(e_\mu - \mathrm{i}e_{n+\mu})/\sqrt{2}. \tag{4.1}$$

Then $\{k, \ell, e_\alpha\}(\alpha \neq \mu, n+\mu)$ gives a null orthonormal frame for (1.1) or (1.2):

$$\begin{aligned} &\langle k, k\rangle = \langle \ell, \ell\rangle = 0, && \langle k, \ell\rangle = 1, \\ &\langle k, e_\alpha\rangle = \langle \ell, e_\alpha\rangle = 0, && \langle e_\alpha, e_\beta\rangle = \delta_{\alpha,\beta}. \end{aligned} \tag{4.2}$$

The covariant derivatives are given by $\nabla_{e_b} e_a = \omega_{ca}(e_b)e_c$, which can be easily calculated by (2.3) and (3.2). Especially, we have

$$\nabla_k k = 0, \tag{4.3}$$

which means that the integral curve of k is a geodesic. It is easy to confirm that the Weyl curvature satisfies the type D condition:

$$\begin{gathered} W(k, e_\alpha, e_\beta, e_\gamma) = W(\ell, e_\alpha, e_\beta, e_\gamma) = 0, \qquad W(k, e_\alpha, k, e_\beta) = W(\ell, e_\alpha, \ell, e_\beta) = 0, \\ W(k, \ell, k, e_\alpha) = W(k, \ell, \ell, e_\alpha) = 0. \end{gathered} \tag{4.4}$$

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